The Coin Toss Space

Tianshu Huang

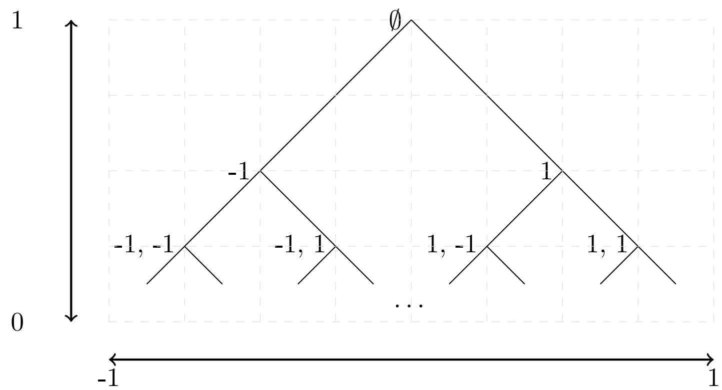
September 17, 2018

1 Introduction

The coin toss space $\mathcal{C}$ is defined as a set of all sequences $\{-1, 1\}^{\mathbb{N}}$, where for two points $x, y \in \mathcal{C}$, $d(x, y) = 2^{-i(x,y)}$, where $i(x, y) = \inf\{i \in \mathbb{N} : x_i \neq y_i\}$ (the first place where x and y are different).

2 Visualizing the coin-toss space

Visualize the coin-toss space as a binary tree on the number line. If the next element is 1, move to the right on the binary tree; if the next element is -1, move left. Each level takes up half of the vertical height of the previous. Then, the distance between two branches is the vertical coordinate where they first diverge.



3 Balls

For any ε, first find $n : 2^{-n} \leq \varepsilon$. Then, to find $B_\varepsilon(x)$, follow the branch represented by x down n levels. All $y \in \mathcal{C}$ that start with that branch are members of that ball.

4 Cylinders

A cylinder in the coin-flip space is generated by fixing some finite number of coin flips. Since balls are generated by fixing the first n coin flips, balls are therefore cylinders.

5 Relationship with $[-1, 1] \in \mathbb{R}$

Now, map each $x = \{x_1, x_2, ...\}$ to $\mathbb{R}$ by starting at 0 and moving left by 2^{i-1} if the next term is -1 and right by 2^{i-1} if the next term is 1. Then, $x \to \sum_i x_i 2^{i-1}$.