Jensen's Inequality *Let $m(\Omega) = 1$. If $f \in L^1(\mu), a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then*

$$\varphi\left(\int_\Omega f d\mu\right) \leq \int_\Omega (\varphi \circ f) d\mu$$

Define $t = \displaystyle\int_\Omega f d\mu$. $a < f(x) < b \Rightarrow a < t < b$.

Let $\beta = \sup_{s<t}\left(\dfrac{\varphi(s) - \varphi(t)}{s - t}\right)$. Since φ is convex, for $a < s < t < u < b$, $\dfrac{\varphi(s) - \varphi(t)}{s - t} \leq \beta \leq \dfrac{\varphi(u) - \varphi(t)}{u - t}$. Note that β could also be set to the inf of the right side.

For a given x if $f(x) < t$, substitute $s = f(x)$:

$$\frac{\varphi(f(x)) - \varphi(t)}{f(x) - t} \leq \beta \Rightarrow \varphi(f(x)) - \varphi(t) \geq \beta(f(x) - t)$$

Since $f(x) - t < 0$. If $f(x) \geq t$, use $u = f(x)$:

$$\frac{\varphi(f(x)) - \varphi(t)}{f(x) - t} \geq \beta \Rightarrow \varphi(f(x)) - \varphi(t) \geq \beta(f(x) - t)$$

In both cases, we have $\varphi(f(x)) - \varphi(t) - \beta(f(x) - t) \geq 0$. Then, integrate this expression, to obtain the inequality

$$\begin{aligned}
\int_\Omega \varphi(f(x)) - \varphi(t) - \beta(f(x) - t) d\mu \geq 0 \\
\int_\Omega \varphi(f(x)) d\mu &\geq \int_\Omega \varphi(t) d\mu + \int_\Omega \beta f(x) d\mu - \int_\Omega \beta t \, d\mu \\
\int_\Omega \varphi(f(x)) d\mu &\geq \varphi(t) + \beta \int_\Omega f(x) d\mu - \beta t \\
\int_\Omega \varphi(f(x)) d\mu &\geq \varphi\left(\int_\Omega f(x)\right)
\end{aligned}$$